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ON THE ROTATIONAL MOTION OF A SOLID CARRYING A VISCO-ELASTIC DISC IN A CENTRAL FIELD OF FORCE*

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The motion of a mechanical system consisting of a symmetrical solid and a round plate (disc) located in the equatorial plane of the ellipsoid of inertial of the solid is considered. It is assumed that the centre of mass of the system moves in a circular orbit in a Newtonian field of force. The disc flexural deformation, accompanied by the dissipation of energy, are the cause of the development of rotational motion in the system. Approximate equations that define this development are obtained, using the method of motion separation and of averaging /1—3/. The averaged equations that define the evolution in Andoyer variables are similar to the equations that describe the evolution of motions of a satellite with flexible visco-elastic rods located along the axis of symmetry of the satellite /3/.

Let the system of equations $Cx_1x_2x_3$ be rigidly attached to a symmetric solid Cx_3 is the axis of symmetry), and let a disc be located in the plane Cx_1x_2. The radius vector of any point on the disc is defined by

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + w\mathbf{e}_3,\quad x_1 = r\cos\theta,\quad x_2 = r\sin\theta \qquad (1)$$
$$0 \leqslant r \leqslant a,\ 0 \leqslant \theta \leqslant 2\pi,\ (x_1, x_2) \in \Omega = \{x_1^2 + x_2^2 \leqslant a^2\}$$

where $w(r, \theta, t)$ is the displacement of points of the elastic disc along the axis Cx_3 during bending, $\mathbf{e}_i$ $(i = 1, 2, 3)$ is the unit vector of the axis Cx_i, and r, θ are the polar coordinates in the region Ω.

Consider the problem when the centre of mass C describes around the attracting centre O a circular Keplerian orbit of radius R and the bending oscillations of the disc do not affect its motion. We introduce the system of coordinates $C\xi_1\xi_2\xi_3$, moving translationally, and the $C\xi_3$ axis is orthogonal to the plane of the orbit. The radius vector of the centre of attraction has in system $C\xi_1\xi_2\xi_3$ the projections $(R\cos\omega_0 t, R\sin\omega_0 t, 0)$, where ω_0 is the orbital angular velocity. Let μ be the gravitational constant of the Newtonian field; then $\omega_0^2 = \mu R^{-3}$.

We will henceforth assume that the description of the deformed state of the disc conforms to the usual assumptions of the linear theory of small deflections of thin plates. In particular, when considering the deflection of a disc of constant rigidity D, the potential energy functionals of elastic deformations and of dissipative forces are defined by the formulae /4/

$$E[w] = \frac{D}{2}\iint_\Omega \Big\{(\Delta w)^2 - 2(1-\nu)\Big[\frac{\partial^2 w}{\partial r^2}\Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\Big) - \qquad (2)$$
$$\frac{1}{r^2}\Big(\frac{\partial^2 w}{\partial r\,\partial\theta} - \frac{1}{r}\frac{\partial w}{\partial\theta}\Big)^2\Big]\Big\} r\,dr\,d\theta$$
$$D[w^{\boldsymbol{\cdot}}] = \chi E[w^{\boldsymbol{\cdot}}],\quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},\quad D = \frac{Eh^3}{12(1-\nu^2)}$$

where Δ is the Laplace operator, D is the bending rigidity of the disc, and E, ν are the modulus of elasticity and Poisson's ratio of the material, respectively, h is the disc thickness, assumed constant, and χ is a coefficient that takes into account the dissipation of energy of deformation. The region of definition of the above functionals (2) is the Sobolev space $W_2^2(\Omega)$. The second relation in (2) assumes that the dissipative functional $D[w^{\boldsymbol{\cdot}}]$ is

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proportional to the potential energy of the elastic deformation, if in the latter the components of the small-deformation tensor are replaced by the corresponding components of the deformation-velocity tensor.

For the variables that define the perturbed motion of the system we select the canonical Andoyer variables I_k, φ_k $(k = 1, 2, 3)$ /3/. When constructing the averaged equations it is convenient to use the Routh functional $R_*[\mathbf{I}, \varphi, w^\cdot, w]$ and the corresponding Routh Eq.(2).

$$I_k^\cdot = -\nabla_{\varphi_k} R_*, \quad \varphi_k^\cdot = \nabla_{I_k} R_* \quad (k = 1, 2, 3) \tag{3}$$
$$d/dt \nabla_{w^\cdot} R_* - \nabla_w R_* = -Q_w$$

where ∇ is the gradient of the functional R_* with respect to the corresponding variable, and Q_w are dissipative forces. Usually $Q_w = -\nabla_{w^\cdot} D[w^\cdot]$, where $D[w^\cdot]$ is the dissipative functional defined in (2). The Routh functional in Andoyer variables has the form

$$R_* = \frac{1}{2}(\mathrm{G} - \mathrm{G}_w, J^{-1}[w](\mathrm{G} - \mathrm{G}_w)) - \frac{1}{2}\int_\Omega w^{\cdot 2}\rho\, dx + \Pi[w] + E[w], \quad dx = dx_1\, dx_2 \tag{4}$$

where $\Pi[w]$ is the potential of the graviation forces and of the forces of inertia of the transfer motion acting on points of the system, and

$$\Pi = -\int_V \{[\mathrm{R} + O(\mathbf{r} + \mathbf{w})]^2\}^{-1/2}\mu\rho\, dx - \frac{1}{2}\omega_0^2 \int_V [\mathrm{R} + O(\mathbf{r} + \mathbf{w})]^2 \rho\, dx \tag{5}$$

where V is the region occupied by the solid and disc $\mathbf{w} = w\mathbf{e}_3$. When integration is carried out with respect to points of the solid, then $\mathbf{w} \equiv 0$ and ρ is the density of the solid, and in the integration over the region Ω occupied by the disc $\mathbf{w} \neq 0$ and ρ is the disc density.

The inertia tensor of the system in the coordinate system $Cx_1x_2x_3$ has the form

$$J[w] = J_0 + J_1[w] + J_2[w]$$
$$J_0 = \operatorname{diag}\{A, A, C\}, \ J_1[w] = \| J_{ij}^{(1)} \|$$
$$J_{ii}^{(1)} = J_{12}^{(1)} = J_{21}^{(1)} = 0, \ i = 1, 2, 3$$
$$J_{13}^{(1)} = J_{31}^{(1)} = -\iint_\Omega \rho w r^2 \cos\theta\, dr\, d\theta$$
$$J_{23}^{(1)} = J_{32}^{(1)} = -\iint_\Omega \rho w r^2 \sin\theta\, dr\, d\theta$$

where $J_1[w]$ and $J_2[w]$ are the components of the inertia tensor, linear and quadratic in w We will henceforth represent $J[w]$ in the form

$$J^{-1}[w] = J_0^{-1} - J_0^{-1} J_1[w] J_0^{-1} + \ \ldots \tag{6}$$

and confine ourselves in series (6) to the first two terms.

The orthogonal matrix $O(t) \in SO(3)$ in Π which determines the transition from the system of coordinates $Cx_1x_2x_3$ to that of Koenig coordinates $C\xi_1\xi_2\xi_3$, and the vector of the angular momentum G is expressed in terms of Andoyer variables

$$\mathrm{G} = \left(\sqrt{I_2^2 - I_1^2}\sin\varphi_1, \ \sqrt{I_2^2 - I_1^2}\cos\varphi_1, \ I_1\right)$$
$$\mathrm{G}_w = \iint_\Omega ((\mathbf{r} + \mathbf{w}) \times \mathbf{w}^\cdot)\rho r\, dr\, d\theta = \iint_\Omega (\mathbf{r} \times \mathbf{w}^\cdot)\rho r\, dr\, d\theta$$
$$O(t) = \Gamma_3(\varphi_3)\Gamma_1(\delta_1)\Gamma_3(\varphi_2)\Gamma_1(\delta_2)\Gamma_3(\varphi_1)$$
$$\Gamma_3(\beta) = \begin{Vmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad \Gamma_1(\beta) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{Vmatrix}$$

Since $R \gg |\mathbf{r} + \mathbf{w}|$, the integrand of the first integral (5) can be expanded in series. Limiting the series to quadratic terms in $|\mathbf{r} + \mathbf{w}|R^{-1}$, taking into account $\mu R^{-3} = \omega_0^2$ and neglecting unimportant constants, we obtain

$$\Pi = -\frac{3}{2}\omega_0^2 \int_V (O^{-1}\mathrm{R}^\circ, \ \mathbf{r} + \mathbf{w})^2 \rho\, dx, \quad \mathrm{R}^\circ = \frac{\mathrm{R}}{|\mathrm{R}|} \tag{7}$$

In the system of coordinates $Cx_1x_2x_3$ the components of the vector $O^{-1}\mathrm{R}^0$ $(\gamma_1, \gamma_2, \gamma_3)$ are /3/

$$\gamma_1 = (\cos\alpha\cos\varphi_3 + \sin\alpha\cos\delta_1\sin\varphi_3)\cos\varphi_1 +$$
$$[(-\cos\alpha\sin\varphi_3 + \sin\alpha\cos\delta_1\cos\varphi_2)\cos\delta_2 -$$
$$\sin\alpha\sin\delta_1\sin\delta_2]\sin\varphi_1$$

$$\gamma_2 = -(\cos\alpha\cos\varphi_2 + \sin\alpha\cos\delta_1\sin\varphi_2)\sin\varphi_1 + [(-\cos\alpha\sin\varphi_2 + \sin\alpha\cos\delta_1\cos\varphi_2)\cos\delta_2 - \sin\alpha\sin\delta_1\sin\delta_2]\cos\varphi_1$$

$$\gamma_3 = (\cos\alpha\sin\varphi_2 - \sin\alpha\cos\delta_1\cos\varphi_2)\sin\delta_2 - \sin\alpha\sin\delta_1\cos\delta_2$$

$$\cos\delta_1 = I_3/I_2,\ \cos\delta_2 = I_1/I_2,\ \alpha = \omega_0 t - \varphi_3$$

The second equation of system (3) is the differential equation of bending oscillations of the disc, which can be represented in the form

$$\rho\frac{\partial^2 w}{\partial t^2} + D\left(1 + \chi\frac{\partial}{\partial t}\right)\Delta\Delta w + \frac{d}{dt}[J^{-1}[w](\mathbf{G} - \mathbf{G}_w)\times\mathbf{r}]\rho\mathbf{e}_3 + \frac{1}{2}(\mathbf{G} - \mathbf{G}_w,\ \nabla_w J^{-1}[w](\mathbf{G} - \mathbf{G}_w)) - 3\omega_0^2\rho(O^{-1}\mathbf{R}^\circ,\ \mathbf{r} + \mathbf{w})(O^{-1}\mathbf{R}^\circ,\ \mathbf{e}_3) = 0 \tag{8}$$

It is necessary to supplement (8) by the boundary conditions of the problem which follow from the expression for the variation of the potential energy of the elastic deformations of the disc (2) /4/

$$\delta E = D\iint_\Omega \Delta^2 w\delta w r\,dr\,d\theta - \oint_{\delta\Omega} M_r\delta\left(\frac{\partial w}{\partial r}\right) r\,d\theta + \oint_{\partial\Omega}\left(N_r - \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}\right)\delta w r\,d\theta \tag{9}$$

Here M_r, $M_{r\theta}$ are the bending moment and torque, respectively, and N_r is transverse force, which on the disc contour have the form

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]$$

$$M_{r\theta} = D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right),\quad N_r = -D\frac{\partial(\Delta w)}{\partial r}$$

For rigid fastening the disc edges, the variations δw and $\delta(\partial w/\partial r)$ are zero and the contour integrals in (9) vanish. The boundary conditions in that case have the simple form

$$w|_{r=a} = 0,\quad \left.\frac{\partial w}{\partial r}\right|_{r=a} = 0 \tag{10}$$

The first condition shows that the disc edges do not have a vertical displacement on deformation, and the second that the slope of the bent surface of the disc at the edge is zero.

The problem considered here contains a "large" parameter that is a characteristic of the rigidity of an elastic disc (the disc rigidity is assumed to be large and its transverse deformations small). In the limit when the rigidity is infinitely large, the bending deformations of the disc are zero ($w \equiv 0$) and if we set the second small parameter, the angular velocity of the orbital system of coordinates ω_0, equal to zero, the motion of the system becomes a regular precession, since then the solid obtained is symmetric ($A = B \neq C$). We assume the motion to be unperturbed. The Routh function and the canonical equations of motion of the solid in Andoyer variables have the form

$$R_0 = \frac{I_2^2 - I_1^2}{2A} + \frac{I_1^2}{2C},\quad I_i^\cdot = 0\quad (i = 1,\ 2,\ 3)$$

$$\varphi_1^\cdot = \frac{A-C}{AC}I_1,\quad \varphi_2^\cdot = \frac{I_2}{A},\quad \varphi_3^\cdot = 0 \tag{11}$$

According to the method of separation of motions it is necessary to find the solution of (8) with the condition that the canonical Andoyer variables correspond to the unperturbed problem (11). We will seek this solution in the form of a series in the small parameter $\varepsilon = D^{-1}$

$$w = \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \tag{12}$$

It is sufficient to determine w_1 since subsequently we propose applying the method of averaging to Eqs.(3) /2/. We will seek the function $w_1(r, t)$ in the form

$$w_1(\mathbf{r},\ t) = \sum_{n=0}^{\infty}(-\chi)^n\frac{\partial^n w_{10}(\mathbf{r},\ t)}{\partial t^n} \tag{13}$$

where $w_{10}(\mathbf{r}, t)$ is the solution of (8) when $\chi = 0$ and with the condition for preserving the terms in accordance with (11) that contain the small parameter ε to the zeroth power. We have

$$\Delta\Delta w_{10} + \rho(J_0^{-1}\mathbf{G}^\cdot\times\mathbf{r})\mathbf{e}_3 - \frac{1}{2}(J_0^{-1}\mathbf{G},\ \nabla_w J_1[w]J_0^{-1}\mathbf{G}) - 3\omega_0^2\rho(O^{-1}\mathbf{R}^\circ,\ \mathbf{r})(O^{-1}\mathbf{R}^\circ,\ \mathbf{e}_3) = 0 \tag{14}$$

Note that

$$\rho(J_0^{-1}G^{\cdot} \times r)e_3 - \frac{1}{2}(J_0^{-1}G, \nabla_w J_1[w] J_0^{-1}G) =$$
$$g(r\sin\varphi_1\cos\theta + r\cos\varphi_1\sin\theta),$$
$$g = \rho A^{-2}C^{-1}(2A - C)I_1\sqrt{I_2^2 - I_1^2}$$

As the result Eq.(14) takes the form

$$\Delta\Delta w_{10} = g_1(t)\, r\cos\theta + g_2(t)\, r\sin\theta$$
$$g_1(t) = -g\sin\varphi_1 + 3\omega_0^2\rho\gamma_1\gamma_3,\quad g_2(t) = -g\cos\varphi_1 + 3\omega_0^2\rho\gamma_2\gamma_3$$

By satisfying the boundary conditions (10), we obtain its solution in the form /5/

$$w_{10} = \frac{r(a^2 - r^2)^2}{192}(g_1(t)\cos\theta + g_2(t)\sin\theta) \tag{15}$$

The convergence of series (13) depends on $\chi\varphi^{\cdot}$, where $\varphi^{\cdot} = \max(|\varphi_1^{\cdot}|, |\varphi_2^{\cdot}|, \omega_0)$. Assuming the quantity $\chi\varphi^{\cdot}$ to be fairly small, $(\chi\varphi^{\cdot} \ll 1)$, and seeking to obtain qualitative results, we limit (13) to the first two terms, assuming

$$w_1(\mathbf{r}, t) = w_{10}(\mathbf{r}, t) - \chi w_{10}^{\cdot}(\mathbf{r}, t) \tag{16}$$

By taking the displacements $w_1(\mathbf{r}, t)$ in (3), we obtain the following equations for the perturbed problem:

$$I_1^{\cdot} = -\nabla_{\varphi_1}R_* = -\left(J^{-1}[w](G - G_w), \frac{\partial G}{\partial\varphi_1}\right) + \tag{17}$$
$$3\omega_0^2\left[(A - C)\gamma_3\frac{\partial\gamma_3}{\partial\varphi_1} + \iint_\Omega r\varepsilon(w_{10} - \chi w_{10}^{\cdot})\frac{\partial}{\partial\varphi_1}(\gamma_3\gamma_1\cos\theta +\right.$$
$$\gamma_3\gamma_2\sin\theta)\rho r\, dr\, d\theta$$
$$I_2^{\cdot} = -\nabla_{\varphi_2}R_* = 3\omega_0^2\left[(A - C)\gamma_3\frac{\partial\gamma_3}{\partial\varphi_2} + \right.$$
$$\left.\iint_\Omega r\varepsilon(w_{10} - \chi w_{10}^{\cdot})\frac{\partial}{\partial\varphi_2}(\gamma_3\gamma_1\cos\theta + \gamma_3\gamma_2\sin\theta)\rho r\, dr\, d\theta\right]$$
$$I_3^{\cdot} = -\nabla_{\varphi_3}R_* = -3\omega_0^2\left[(A - C)\gamma_3\frac{\partial\gamma_3}{\partial\alpha} + \right.$$
$$\left.\iint_\Omega r\varepsilon(w_{10} - \chi w_{10}^{\cdot})\frac{\partial}{\partial\alpha}(\gamma_3\gamma_1\cos\theta + \gamma_3\gamma_2\sin\theta)\rho r\, dr\, d\theta\right]$$

Let us average the right sides of (17) over the fast angular variables $\varphi_1, \varphi_2, \alpha$. The averaged equations are very cumbersome and are difficult to analyse. However, the varaibles "action" evolve at different velocities. The right side of the first equation of (17) contains, after averaging terms proportional to $\chi\varepsilon$ and $\chi\varepsilon\omega_0^2$, while the last two equations of (17) contain terms proportional to $\chi\varepsilon\omega_0^2$. If ω_0 is fairly small, the evolution of the action variables can be divided into two stages. In the first stage of "rapid" evolution of the variables we assume $\omega_0 = 0$. This means that in the evolution process, the variables I_2 and I_3 are constant, and only the variable I_1 changes. At the second stage of slow evoltuion we assume that the rapid evolution has been completed (the variable I_1 takes its limit values), and only the evolution of the variables I_2 and I_3 is considered. Assuming that in (17) $\omega_0^2 = 0$ (the system moves by inertia), we obtain for the variable I_1 the equation

$$I_1^{\cdot} = -\nabla_{\varphi_1}R_* = -\left(J^{-1}[w](G - G_w), \frac{\partial G}{\partial\varphi_1}\right) \tag{18}$$

where the accuracy of small ε^2 we have

$$(J^{-1}[w](G - G_w), \frac{\partial G}{\partial\varphi_1}) \cong \left(J_0^{-1}G, \frac{\partial G}{\partial\varphi_1}\right) -$$
$$\left(J_0^{-1}J_1[w]J_0^{-1}G, \frac{\partial G}{\partial\varphi_1}\right) - \left(\iint_\Omega[\mathbf{r}\times\mathbf{w}^{\cdot}]\rho r\, dr\, d\theta, J_0^{-1}\frac{\partial G}{\partial\varphi_1}\right)$$

Denoting the averaging over the angles φ_1, α by the symbol $\langle\cdot\rangle$, from (18) we obtain

$$I_1^{\cdot} = A^{-1}C^{-1}I_1\sqrt{I_2^2 - I_1^2}\langle J_{13}^{(1)}\cos\varphi_1 - J_{23}^{(1)}\sin\varphi_1\rangle + \tag{19}$$
$$\rho A^{-1}\sqrt{I_2^2 - I_1^2}\left\langle\iint_\Omega w^{\cdot}(r\sin\varphi_1\cos\theta + r\cos\varphi_1\sin\theta)r\, dr\, d\theta\right\rangle$$
$$\langle\cdot\rangle_{\varphi_1,\alpha} = (2\pi)^{-2}\int_0^{2\pi}\int_0^{2\pi}\cdot\, d\varphi_1\, d\alpha$$

Averaging of the right-hand side of (19), we obtain

$$I_1^{\cdot} = -n_1 I_1^3 (I_2^2 - I_1^2), \tag{20}$$
$$n_1 = \varepsilon\chi\rho^2 \frac{(2A-C)^2(A-C)}{A^6C^3}\pi\int_0^a \frac{(r^2-a^2)^2}{192} r^3\,dr$$

The sign of n_1 is the same as that of $A - C$. When $A > C, I_1$ approaches zero, and when $A < C\ I_1$ approaches I_2. This means that regular precession ends in rotation around the angular vector, which lies either in the equatorial plane of the ellipsoid of inertia or coincides with its axis of inertia.

We will consider the case when $A > C$, and we will determine the evolution of the variables I_2 and I_3. In accordance with the above, we assume the evolution of the variable I_1 to be completed and set $I_1 = 0$. The remaining two equations of system (17) now take the form

$$I_2^{\cdot} = 9\omega_0^4\rho^2 p\varepsilon\left\langle\sum_{i=1}^{2}\left[\gamma_3\gamma_i - \chi\frac{d}{dt}(\gamma_3\gamma_i)\right]\frac{\partial}{\partial\varphi_2}(\gamma_3\gamma_i)\right\rangle \tag{21}$$
$$I_3^{\cdot} = -9\omega_0^4\rho^2 p\varepsilon\left\langle\sum_{i=1}^{2}\left[\gamma_3\gamma_i - \chi\frac{d}{dt}(\gamma_3\gamma_i)\right]\frac{\partial}{\partial\alpha}(\gamma_3\gamma_i)\right\rangle$$
$$p = \pi\int_0^a \frac{(a^2-r^2)^2}{192} r^3\,dr$$

In Eqs.(21) averaging is carried out over the variables φ_2 and α, and the γ_i when $I_1 = 0$, are

$$\gamma_1 = \cos\alpha\cos\varphi_2 + \sin\alpha\cos\delta_1\sin\varphi_2,\ \gamma_2 = -\sin\alpha\sin\delta_1 \tag{22}$$
$$\gamma_3 = \cos\alpha\sin\varphi_2 - \sin\alpha\cos\varphi_2\cos\delta_1$$

When calculating (22) φ_1 was taken as zero, which does not affect the generality. Averaging of the right-hand sides of Eqs.(21) yields

$$I_2^{\cdot} = -n_2\left[8A\omega_0 I_3 I_2^{-1} - 4I_2(1 + I_3^2 I_2^{-2})\right] \tag{23}$$
$$I_3^{\cdot} = n_2\left[A\omega_0(5 + 6I_3^2I_2^{-2} - 3I_3^4I_2^{-4}) - 8I_3\right]$$
$$n_2 = (9/16)\omega_0^4\rho^2\chi p\varepsilon A^{-1} > 0$$

Equating the right-hand sides of (23) to zero, wer obtain the stationary points that are solutions corresponding to constant I_2 and I_3. We have

$$3\lambda^6 - 3\lambda^4 + 5\lambda^2 - 5 = 0,\ \lambda = I_3I_2^{-1},\ |\lambda| \leqslant 1 \tag{24}$$

When $|\lambda| \leqslant 1$ this equation has a unique solution $\lambda = 1$, corresponding to the steady solution $I_2 = I_3 = A\omega_0$, which defines the uniform rotation of the system with angular velocity ω_0 about the axis $C\xi_3$. The stability of this steady point follows from the equations in variations of the form

$$\xi^{\cdot} = -8n_2\xi,\ \eta^{\cdot} = -8n_2\eta,\ \xi = I_2A^{-1}\omega_0^{-1} - 1,$$
$$\eta = I_3A^{-1}\omega_0^{-1} - 1$$

For steady rotation with $I_2 = I_3 = A\omega_0$ the disc is fixed in the orbital system of coordinates. To determine its position in the orbital system of coordinates we introduce the angle β between the unit vector $\mathbf{R}^0$ and the axis of symmetry. We have

$$\beta = \omega_0 t + \pi/2 - \varphi_2 - \varphi_3$$
$$\beta^{\cdot} = \omega_0 - \varphi_2^{\cdot} - \varphi_3^{\cdot} = \omega_0 - \nabla_{I_2}R_* - \nabla_{I_3}R_*$$
$$\nabla_{I_2}R_* = \left(J^{-1}[w](\mathrm{G} - \mathrm{G}_w), \frac{\partial \mathrm{G}}{\partial I_2}\right) - 3\omega_0^2\left[(A-C)\gamma_3\frac{\partial\gamma_3}{\partial I_2} + \iint_\Omega\sum_{i=1}^{2} wx_i\frac{\partial}{\partial I_2}(\gamma_3\gamma_i)\rho\,dx\right]$$
$$\nabla_{I_3}R_* = -3\omega_0^2\left[(A-C)\gamma_3\frac{\partial\gamma_3}{\partial I_3} + \iint_\Omega\sum_{i=1}^{3} wx_i\frac{\partial}{\partial I_3}(\gamma_3\gamma_i)\rho\,dx\right]$$

which after calculations gives

$$\beta^{\cdot} = -A^{-1}I_2 + \omega_0 + {}^3/_2A^{-1}\omega_0^2\rho^2 p\varepsilon\left[(\sin 2\beta)^{\cdot} - \chi(\sin 2\beta)^{\cdot\cdot}\right]$$

Since $\gamma_1 = \sin\beta,\ \gamma_2 = 0,\ \gamma_3 = \cos\beta,$ we have

$$I_2^{\cdot} = {}^3/_2\omega_0^2(A - C)\sin 2\beta - 9\omega_0^4\rho^2 p\varepsilon\,({}^1/_2\sin 2\beta - \chi\beta^{\cdot}\cos 2\beta)\cos 2\beta$$

and further

$$\beta^{\cdot\cdot} + 9\omega_0^4\rho^2\chi p\varepsilon A^{-1}\beta^{\cdot}\cos^2 2\beta + {}^3/_2\omega_0^2(A - C)A^{-1}\sin 2\beta = 0 \tag{25}$$

where only the principal terms of the corresponding varaibles have been retained. This equation implies that when the motion is steady, the angle β is either $\{\pi k\}_{k=-\infty}^{k=\infty}$, or $\{\pi/2 +$

$\pi k\}_{k=-\infty}^{k=\infty}$. The first set of equilibrium positions is stable (the axis of symmetry of the solid and disc coincide with the radius vector of system), while the second is unstable (the axis of symmetry is tangent to the orbit).

If $A < C$, the evolution of variable I_1 according to (20) leads to $I_1 = I_2$. The evolution of the variables I_2 and I_3 in that case is defined by Eqs.(21) in which

$$\gamma_1 = \cos\alpha\cos\varphi + \sin\alpha\sin\varphi\cos\delta_1,\ \gamma_2 = -\cos\alpha\sin\varphi + \sin\alpha\cos\varphi\cos\delta_1$$
$$\gamma_3 = -\sin\alpha\sin\delta_1,\ \varphi = \varphi_1 + \varphi_2,\ \varphi^{\cdot} = I_2/C$$

Averaging Eqs.(21) over the angles φ and α gives

$$I_2^{\cdot} = -n_3(I_2^2 - I_3^2)I_2^{-2}[I_2(1 + 3I_3^2I_2^{-2}) - 4C\omega_0I_3I_2^{-1}] \tag{26}$$
$$I_3^{\cdot} = -4n_3(I_2^2 - I_3^2)I_2^{-2}[I_3 - C\omega_0(1 + I_3^2I_2^{-2})],\ n_3 = 2AC^{-1}n_2$$

The steady solutions of (26) lie on the straight line $I_2 = I_3$ and the equations in variations in the neighbourhood of the point $I_2 = I_3 = I$ have the form

$$\xi^{\cdot} = -8I^{-1}n_3(I - C\omega_0)(\xi - \eta),\ \eta^{\cdot} = -8I^{-1}n_3(I - 2C\omega_0)(\xi - \eta)$$
$$I_2 = I(1 + \xi),\ I_3 = I(1 + \eta)$$

and further

$$\xi^{\cdot} - \eta^{\cdot} = -8n_3C\omega_0I_1^{-1}(\xi - \eta)$$

This implies that the steady set $I_2 = I_3$ is stable.

This solution corresponds to the rotation of the system around the axis of symmetry which coincides with the normal to the orbit and, since the solid is symmetrical, its angular velocity may not, generally, be the same as the orbital velocity. The disc is such motion is not deformed in the approximation considered above.

Note that the evolution of the rotational motion of a symmetrical solid with an elastic-plastic disc in the equatorial plane of the elliposid of inertia is similar to the evolution of a symmetrical solid with elastic-plastic flexible rods situated on the axis of symmetry of the solid /3/.

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